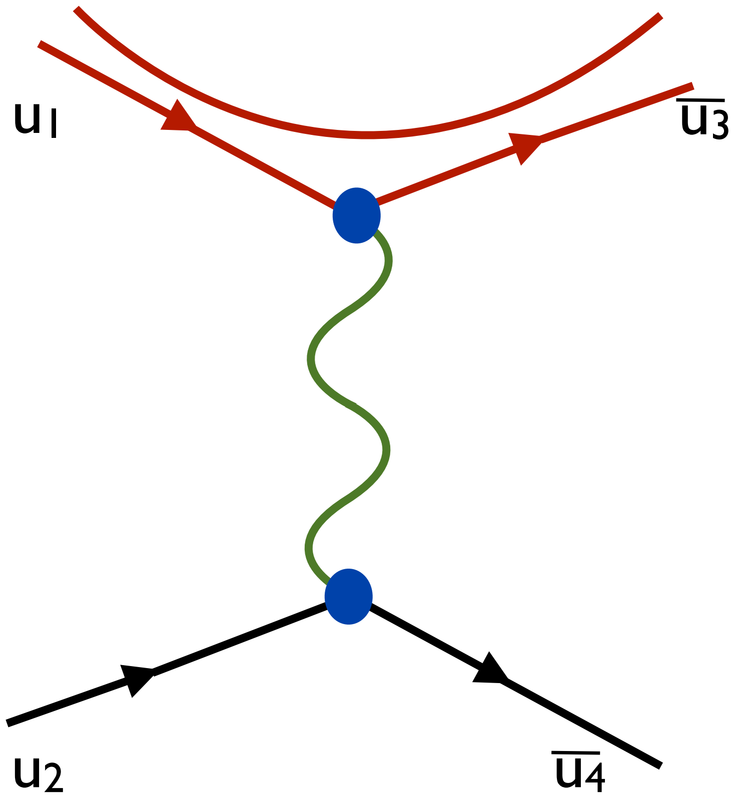
Introduction to Feynman Diagrams

Feynman Diagram Anatomy



Incoming and outgoing particles described by Wavefunctions:

- Spin-0: plane waves
- Spin 1/2: Dirac Spinors
- Spin 1: polarized vectors

Probability current

$$j_\mu = i[\psi^\star \partial_\mu \psi - \psi \partial_\mu \psi^\star] = (\bar{u}_3 \gamma^\mu u_1)$$

Propagators, if q^μ is the energy transferred on propagator:

- Photon is $1/q^2$
- Virtual W/Z boson: $1/(q^2-M_W)$; $1/(q^2-M_Z)$;
- Virtual Fermion of mass m: $(\Upsilon^\mu q_\mu+m)/(q^2-m^2)$

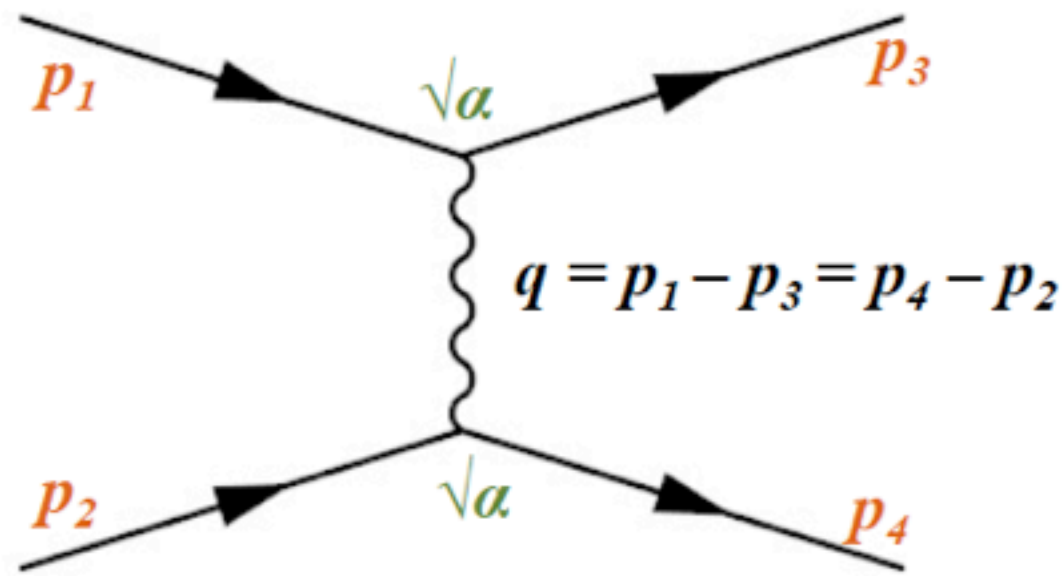
Vertexes have dimensionless coupling constants:

- EM interaction: $\sqrt{\alpha} = e$
- Strong Force: $\sqrt{\alpha_s} = g_s$
- Weak Force: have axial c_A and vector c_V

Each Feynman diagram represent a matrix element (i.e. a process) going from the initial state to the final state, if multiple processes are possible the cross section is proportional to the amplitude of the sum of the processes

Spin-less scattering

The matrix element is calculated by making a product of all the elements in the matrix diagram.
In case of spin-less particle the plane waves reduce to the product of the momenta



Vertex Couplings

$$\mathcal{M} = \frac{\alpha}{q^2}(p_1 + p_3)(p_2 + p_4)\,\delta^4(p_1 + p_2 - p_3 - p_4)$$

Photon Propagator

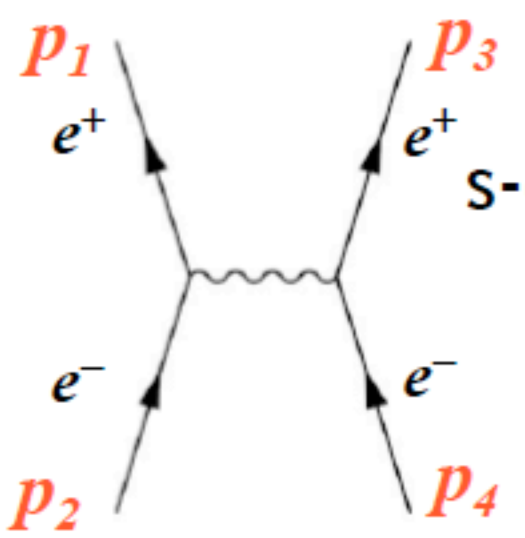
Plane Waves

Four momentum conservation

(in terms of Mandelstam variables) $\mathcal{M} = \dfrac{\alpha\,(s-u)}{t}$

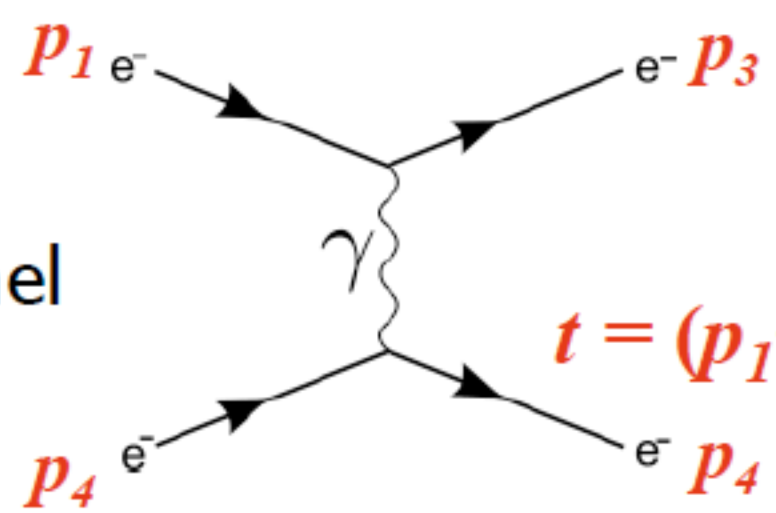
Mandelstam variables

s-channel



$s = (p_1 + p_2)^2$

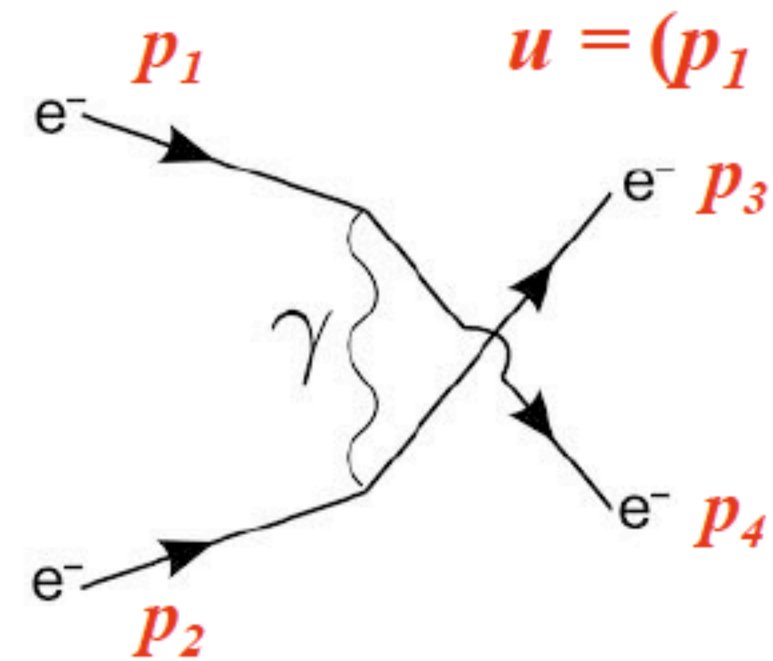
t-channel



$t = (p_1 - p_3)^2$

u-channel

$u = (p_1 - p_4)^2$



For highly relativistic elastic scattering

$p \sim E,\ m \ll E$:

$$s = 4\, p^{*2}$$

$$t = -2\, p^{*2}\, (1 - \cos\theta^*)$$

$$u = -2\, p^{*2}\, (1 + \cos\theta^*)$$

with $p^* = p_1 = p_2$ is the CM momentum of the particles, and θ^* is the CM scattering angle

Electron-Muon scattering

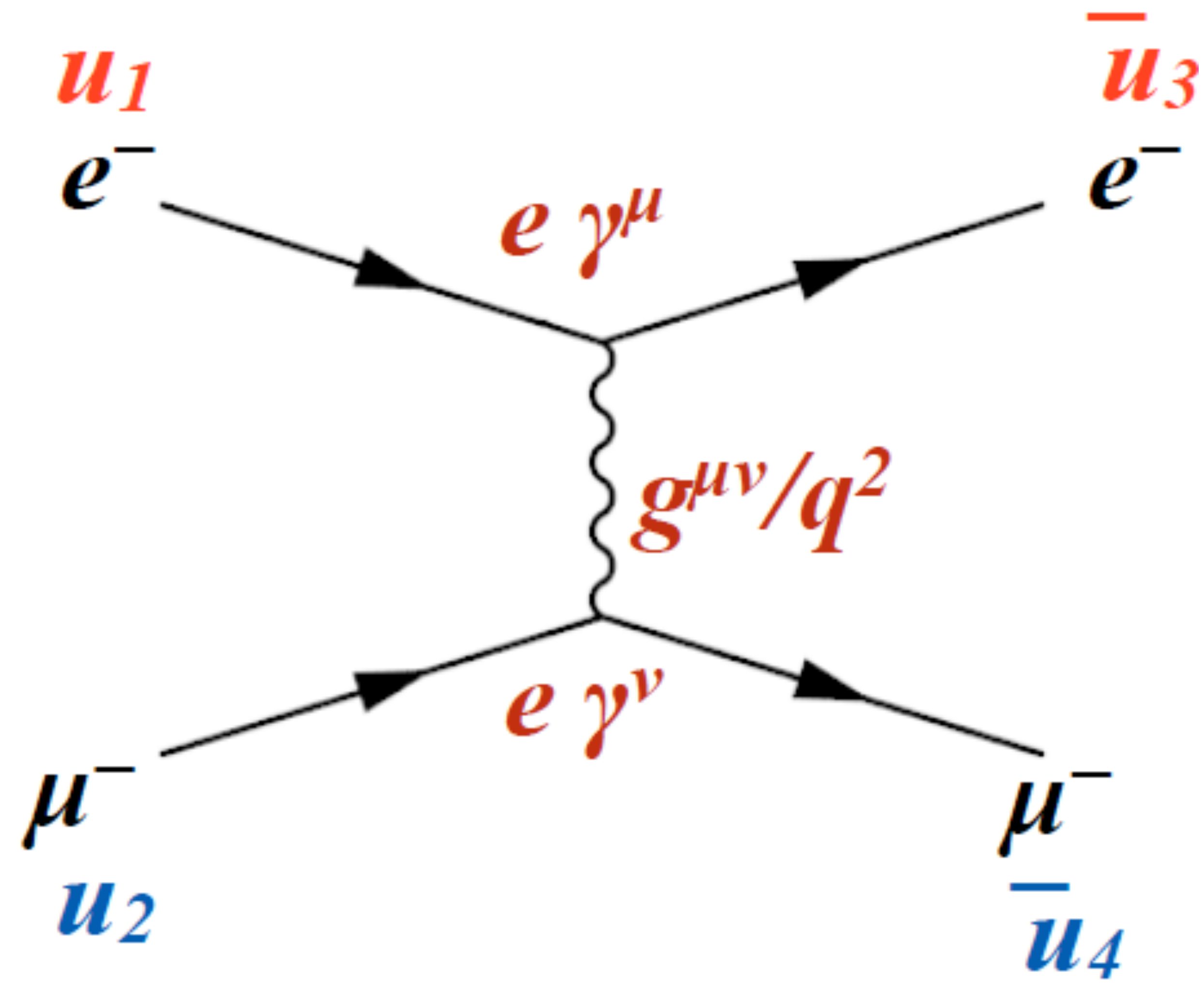
Similar to the previous process but now particles have spin. In this case the current are not planar waves but spinor currents

$$\mathcal{M} = e^2 \frac{g^{\mu\nu}}{q^2} J^{\mu}_{13}\, J^{\nu}_{24}$$

Vertex Couplings

$$\mathcal{M} = e^2 \frac{g^{\mu\nu}}{q^2} \underbrace{(\bar{u}_3 \gamma^{\mu} u_1)}_{\text{Electron current}} \underbrace{(\bar{u}_4 \gamma^{\nu} u_2)}_{\text{Muon current}}$$

Photon propagator,



To obtain the squared matrix element we need to make a product of M with its conjugate

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} (\bar{u}_3 \gamma^{\mu} u_1)\, (\bar{u}_3 \gamma^{\mu} u_1)^* (\bar{u}_4 \gamma^{\mu} u_2) (\bar{u}_4 \gamma^{\mu} u_2)^*$$

Calculating the Matrix element

- let's define the helicity as the projection of the particle spin on the momentum of the particle. This quantity for high energy E>>m is conserved.
- For electron-muon scattering we have only 4 possible contributions

$$\mathcal{M}(e_\uparrow\mu_\downarrow \rightarrow e_\uparrow\mu_\downarrow) + \mathcal{M}(e_\downarrow\mu_\uparrow \rightarrow e_\downarrow\mu_\uparrow) + \mathcal{M}(e_\uparrow\mu_\uparrow \rightarrow e_\uparrow\mu_\uparrow) + \mathcal{M}(e_\downarrow\mu_\downarrow \rightarrow e_\downarrow\mu_\downarrow)$$

- Where the arrows show the helicity of each particle. For calculating the matrix element value we have to sum over the final states and average over initial states

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\frac{1}{(2S_1+1)(2S_2+1)}\sum_{S_3,S_4}(\bar{u}_3\gamma^\mu u_1)(\bar{u}_3\gamma^\nu u_1)^*(\bar{u}_4\gamma^\mu u_2)(\bar{u}_4\gamma^\nu u_2)^*$$

$$= \frac{e^4}{q^4}\left(\frac{1}{(2S_1+1)}\sum_{S_3}(\bar{u}_3\gamma^\mu u_1)(\bar{u}_3\gamma^\nu u_1)^*\right)\left(\frac{1}{(2S_2+2)}\sum_{S_4}(\bar{u}_4\gamma^\mu u_2)(\bar{u}_4\gamma^\nu u_2)^*\right)$$

$$= \frac{e^4}{q^4} L_e\, L_m$$

L_e and L_m still contains the spinors describing the fermion wave functions, in order to calculate the matrix element (and then the cross section, using Fermi Golden Rule) we have to get rid of them

Trace Theorems

There are some theorems (Trace theorems) that allow you to calculate the spinor-gamma matrix products (detailed explanations can be found in most particle physics book e.g. Griffith 7.7). Using these theorems we can compute the values of L_e and L_m

$$\begin{aligned} L_e &= \frac{1}{(2S_1+1)} \sum_{S_3} (\bar{u}_3 \gamma^\mu u_1) (\bar{u}_3 \gamma^\nu u_1)^* \\ &= 2 \left[p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - (p_3 \cdot p_1 - m_e^2) g^{\mu\nu} \right] \end{aligned}$$

$$L_\mu = 2 \left[p_4^\mu p_2^\nu + p_4^\nu p_2^\mu - (p_4 \cdot p_2 - m_\mu^2) g^{\mu\nu} \right]$$

Now L_e and L_m are a function of the quadrimomenta only, $g^{\mu\nu}$ is the operator for making the scalar product out of two quadrivector (i.e. the scalar product bilinear operator in the Minkowski space). From the two quantities we can now go back calculating the matrix element, in the hypothesis that $E \gg m_e$, $E \gg m_\mu$

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} L_e L_\mu = \frac{8e^4}{q^4} \left[(p_3 \cdot p_4)(p_1 \cdot p_2) + (p_3 \cdot p_2)(p_1 \cdot p_4) \right]$$

$$|\mathcal{M}|^2 = 2e^4 \frac{s^2+u^2}{t^2} = 2e^4 \frac{1 + 4\cos^4(\theta^*/2)}{\sin^4(\theta^*/2)}$$

Cross section

We can now calculate the cross section using the Fermi Golden Rule, in this case the density of final states is already incorporated, S is a term taking into account the presence of identical particles in the final states and is 1 if there are not

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1+E_2)^2} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|}$$

Since this is an elastic scattering the initial and final momenta are equal

This is simply reduced to s

This reduces to

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2\pi s}\left(\frac{s^2+u^2}{t^2}\right)$$

We can in principle calculate the value for each contribution to the cross section for each spin state:

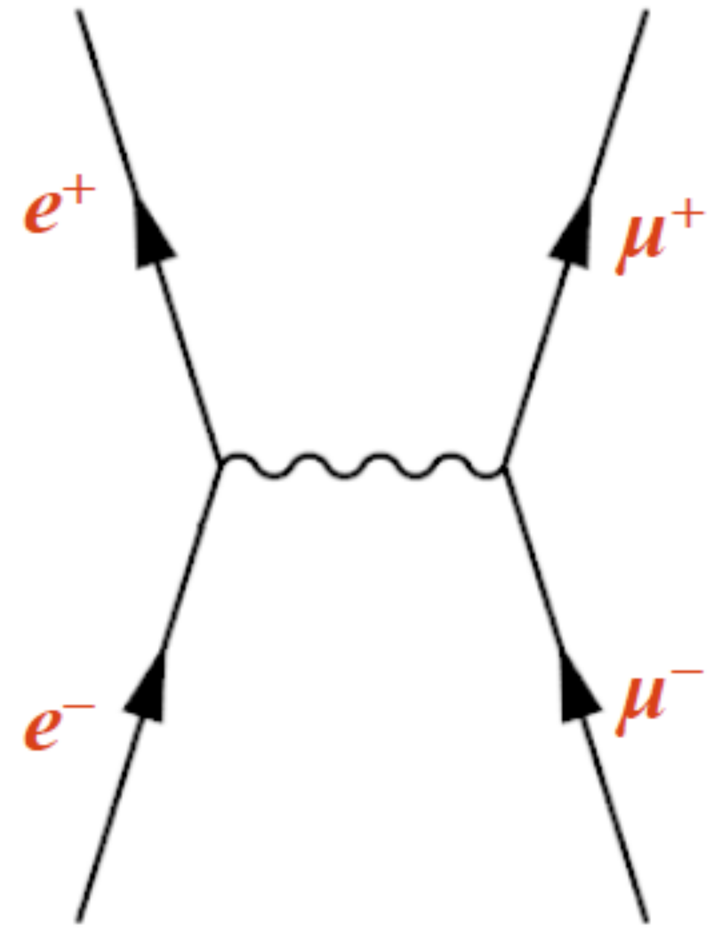
$$\mathcal{M}(\uparrow\uparrow\uparrow\uparrow) = \mathcal{M}(\downarrow\downarrow\downarrow\downarrow) = e^2\frac{u}{t} = e^2\frac{1+\cos\theta^*}{1-\cos\theta^*}$$

$$\mathcal{M}(\uparrow\downarrow\uparrow\downarrow) = \mathcal{M}(\downarrow\uparrow\downarrow\uparrow) = e^2\frac{s}{t} = e^2\frac{2}{1-\cos\theta^*}$$

Where the θ* angle is the angle between the incoming and outgoing electron

$e^+e^- \rightarrow \mu^+\mu^-$ scattering

The diagram for the process is basically the same as for electron muon scattering, we have the same situation exchanging t and s variables.
In fact $p_2 \leftrightarrow -p_3$

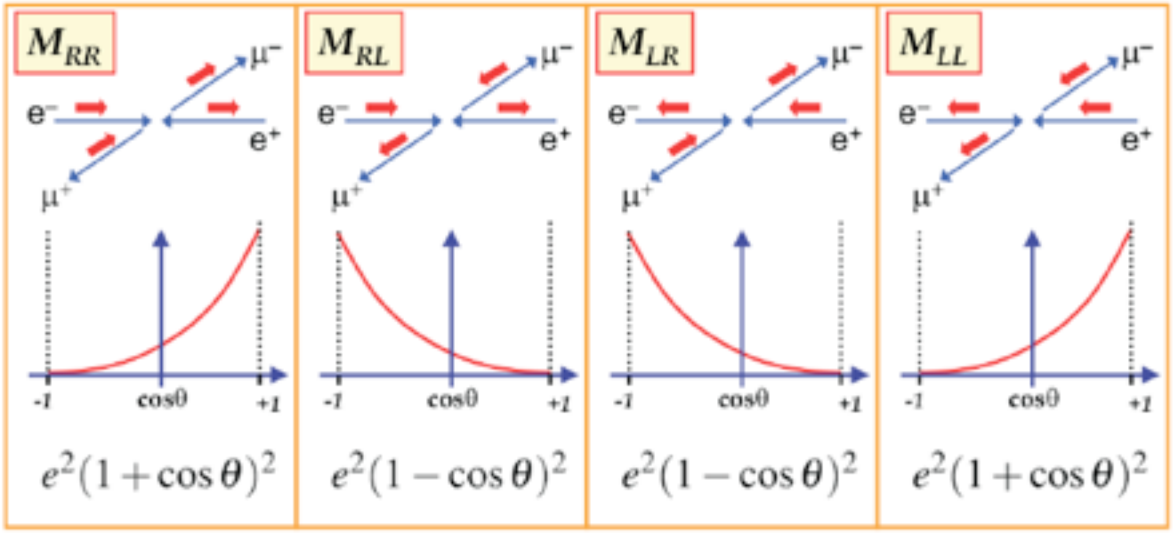


The Matrix element and the cross section can be calculated easily by just exchanging the mandelstam variables

$$|\mathcal{M}|^2 = 2e^4\frac{(t^2+u^2)}{s^2} = e^4(1+\cos^2\theta)$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\pi s}(1+\cos^2\theta)$$

We have still different contributions for different spin states



Total Cross section

- Total cross section, integrate over solid angle:

$$
\begin{aligned}
\sigma &= \int \frac{d\sigma}{d\Omega} d\Omega \\
&= \frac{\alpha^2}{4\pi s} \int (1+\cos^2\theta) d\cos\theta d\phi \\
&= \frac{\alpha^2}{4\pi s} [\phi]_{-\pi}^{\pi} \left[\cos\theta + \frac{1}{3}\cos^3\theta\right]_{\cos\theta=-1}^{\cos\theta=+1} \\
&= \frac{4\alpha^2}{3s}
\end{aligned}
$$

- Comparison prediction to measurement. Pretty good for a 1st order calculation!

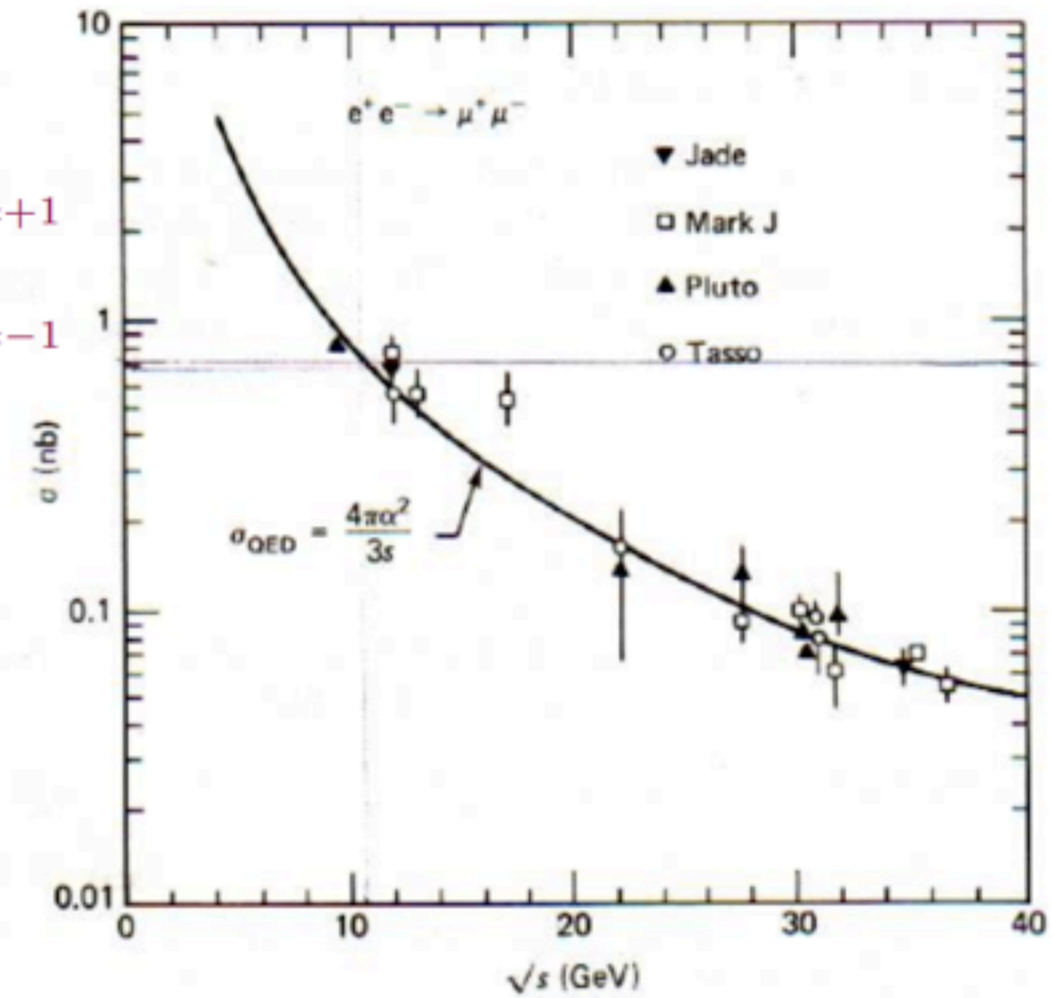


Fig. 6.6 The total cross section for $e^-e^+ \to \mu^-\mu^+$ measured at PETRA versus the center-of-mass energy.